The pure state space of quantum mechanics as Hermitian symmetric space

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Abstract

The pure state space of Quantum Mechanics is investigated as Hermitian Symmetric Kähler manifold. The classical principles of quantum mechanics (Quantum Superposition Principle, Heisenberg Uncertainty Principle, Quantum Probability Principle) and Spectral Theory of observables are discussed in this non-linear geometrical context.
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1. Introduction

Several models of *delinearization of quantum mechanics* have been proposed (see, for instance [2,5,11,16,18,21,23] for a complete list of references). Frequently these proposals are supported by different motivations, but it appears that a common feature is that, more or less, the delinearization must be paid essentially by the superposition principle.

This attitude can be understood if the delinearization program is worked out in the setting of a Hilbert space $\mathcal{H}$ as a ground mathematical structure.

However, as is well known, the ground mathematical structure of QM is the manifold of (pure) states $\mathbf{P}(\mathcal{H})$, the projective space of the Hilbert space $\mathcal{H}$. Since, obviously, $\mathbf{P}(\mathcal{H})$ is

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not a linear object, the popular way of thinking that the superposition principle compels the linearity of the space of states is untenable.

The delinearization program, by itself, is not related in our opinion to attempts to construct a non-linear extension of QM with operators which act non-linearly on the Hilbert space $\mathcal{H}$. The true aim of the delinearization program is to free the mathematical foundations of QM from any reference to linear structure and to linear operators. It appears very gratifying to be aware of how naturally geometric concepts describe the more relevant aspects of ordinary QM, suggesting that the geometric approach could be very useful also in solving open problems in Quantum Theories.

Of course in $\mathbf{P}(\mathcal{H})$ remains of the linearity are well present: one of our aims in this paper is just to show that such remains are represented by the geodesical structure; therefore even the superposition principle *can be delinearized* without affecting its peculiar physical content, suggesting moreover that manifolds of states endowed with a fair geodesical structure could be compatible with the superposition principle. Another feature we stress of our work is that also the spectral theory of observables has a very simple description in terms of the differential structure of $\mathbf{P}(\mathcal{H})$. Indeed, we will show that the usual linear observables are described by functions respecting geodesics in the technical detailed meaning coded in the definition of geolinearity.

A very important bonus of our analysis of observables is the coming out of *suitable classes of non-linear observables*; about this subject we only anticipate a little in this work, because it will be the content of a forthcoming paper [9].

Now, as is well known, $\mathbf{P}(\mathcal{H})$ is a Kähler manifold [11], but the geodesic structure of a Kähler manifold may be very involved [4]. Therefore to look at $\mathbf{P}(\mathcal{H})$ simply as a Kähler manifold could not be the best way to bring into focus the role of the geodesic structure we have stressed above. On the other hand, the geodesic structure is particularly transparent in the subcategory of Hermitian symmetric spaces as one can see, in the finite dimensional case, in the book [19].

In Section 2, we briefly review the Kähler structure of $\mathbf{P}(\mathcal{H})$. Then, we discuss *infinite dimensional* symmetric homogeneous G spaces and their geodesical structure. We prove that $\mathbf{P}(\mathcal{H})$ is a Hermitian symmetric G space with $G = U(\mathcal{H})$. As a bonus we obtain that $\mathbf{P}(\mathcal{H})$ is simply connected, even in the infinite dimensional case.

In Section 3, we carefully discuss the Superposition Principle and show how SP is tied up with geodesic structure of pure state space. In Section 4, observables are characterized in terms of Kähler structure as K functions or, equivalently, in terms of geodesic structure, as geolinear functions.

In Section 5, we discuss Uncertainty Principle in a strong version which holds for Hermitian symmetric G spaces. In Section 6, we discuss spectral theory and quantum probability principle for observables, in a natural geometric way.

2. Projective quantum mechanics and Hermitian *G* spaces

Let us translate standard quantum mechanics (SQM) into geometrical terms, to get projective quantum mechanics [3,6–8,10–14,16].

Pure states in QM are geometrically described as the points of an infinite dimensional Kähler manifold $\mathbf{P}(\mathcal{H})$, the *projective space* of the Hilbert space $\mathcal{H}$ of the system. The

points $\hat{\varphi}$, $\hat{\psi}$, ... of $\mathbf{P}(\mathcal{H})$ (i.e. the rays of $\mathcal{H}$ generated by non-zero vectors φ, ψ of $\mathcal{H}$) are the (pure) states of the quantum system.

$\mathbf{P}(\mathcal{H})$, as a complex manifold, can be canonically regarded as a real smooth manifold with an integrable almost complex structure J. The manifold $(\mathbf{P}(\mathcal{H}), J)$ is endowed with a natural Kählerian metric, i.e. a Riemannian metric g such that

1. $\mathsf{g}_{\hat{\varphi}}(v, w) = \mathsf{g}_{\hat{\varphi}}(Jv, Jw)$, $\quad v, w \in T_{\hat{\varphi}}\mathbf{P}(\mathcal{H})$;
2. the associated fundamental 2-form

$$\omega_{\hat{\varphi}}(v, w) := \mathsf{g}_{\hat{\varphi}}(Jv, w)$$

   is closed, hence symplectic.

The natural Kähler metric of $\mathbf{P}(\mathcal{H})$ is the Fubini–Study metric

$$\mathbf{g}_{\hat{\varphi}}(v, w) = 2\kappa\Re(\mathsf{v}|\mathsf{w}),$$

where $\mathsf{v} = T_{\hat{\varphi}}b_{\varphi}(v)$, $\mathsf{w} = T_{\hat{\varphi}}b_{\varphi}(w)$, and the associated fundamental 2-form

$$\omega_{\hat{\varphi}}(v, w) = 2\kappa\Im(\mathsf{v}, |\mathsf{w}),$$

where $\kappa > 0$ is an (arbitrary) constant. We recall that b_{φ} is the chart at $\hat{\varphi}$ [7]. To get a correct correspondence with ordinary quantum mechanics, one must assume $\kappa = \hbar$.

2.1. Symmetric homogeneous G spaces

Finite dimensional homogeneous G spaces are widely discussed in the literature. Standard reference books are [17,19]. As there are only a few references for the infinite dimensional setting [25], we shortly review definitions and properties in the context of Banach manifolds. The proofs are given only in the case where the extension from finite to the infinite dimensional setting is not easy. By *ordinary Banach manifold*, we mean a second countable connected Hausdorff smooth Banach manifold M. Let G be an ordinary Banach–Lie group acting on M. Then the pair (M, G) will be said to be a *homogeneous G space* if:

1. the action of G on M is smooth and transitive;
2. the isotropy group G_x at x is a Lie subgroup of G for $x \in M$.

Since the mapping $\phi : g \mapsto gx$ of G onto M is continuous, $G_x = \phi^{-1}(x)$ is a closed Lie subgroup of G. Thus G/G_x has a unique smooth (actually, analytic) structure with the property that G/G_x is a G space and the canonical map $\pi : G \to G/G_x$ is smooth (actually, analytic) and open. To prove that the induced surjection $\varphi_x : G/G_x \to M$ is an homeomorphism we have just to prove that it is open. This follows by Theorem A.I.1 in [1]. Arguing as in Proposition 4.3 in [17], we obtain that φ_x is a diffeomorphism.

The symmetric G spaces constitute an important class of homogeneous G spaces. A *symmetric G space* is a triple (M, G, s) where (M, G) is a homogeneous G space and s is an involutive diffeomorphism of M with an isolated fixed point o. Given a symmetric G space (M, G, s) we construct for each point x of the quotient space $M = G/G_o$ an involutive diffeomorphism s_x, called the *symmetry at x*, which has x as isolated fixed point:

for $x = go$, we set $s_x = g \circ s \circ g^{-1}$. Then s_x is independent of the choice of the g such that $x = go$. There is a unique involutive automorphism σ of G such that $\sigma(g) = s(go)$.

In every symmetric G space (M, G, s) one has

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{m} = \{A \in \mathfrak{g} : \sigma_e(A) = -A\}$ (we denote by σ_e the derivative of σ at e) is $Ad(G_x)$ invariant and complements $\mathfrak{h}$ in $\mathfrak{g}$.

A *complex symmetric G space* is a complex Banach manifold M which is also a symmetric G space with holomorphic symmetries and automorphisms. Let M be a real Banach manifold. An *almost complex structure J* on M is a smooth tensor field on M whose value at any point x of M is a complex structure J_x on the tangent space $T_x M$ at x. A smooth map between almost complex manifolds is said to be *almost complex* if its derivative at each point of the domain is complex linear. An almost complex structure J on M is said to be a *complex structure* on M if there exists a smooth almost complex chart at any point $x \in M$. If J is a complex structure on M, then the collection of all such almost complex charts constitutes an atlas on M whose transition functions are holomorphic; we can thus regard M as a complex manifold.

Let us now come to $\mathbf{P}(\mathcal{H})$ and give it the structure of infinite dimensional complex symmetric homogeneous G space. We denote by $U(\mathcal{H})$ the Banach–Lie group of unitary operators of $\mathcal{H}$, by $\mathfrak{u}(\mathcal{H})$ its Lie algebra and by $S^1(\mathcal{H})$ the unit ball of $\mathcal{H}$. The natural action of $U(\mathcal{H})$ on $S^1(\mathcal{H})$ is transitive and quotients to the natural action of $U(\mathcal{H})$ on $\mathbf{P}(\mathcal{H})$.

Proposition 1. *The projective space $\mathbf{P}(\mathcal{H})$ is a complex symmetric G space with automorphism group $G = U(\mathcal{H})$. The scalar product on $\mathfrak{m}$ induces on $\mathbf{P}(\mathcal{H})$ the Fubini–Study metric.*

Proof. For $\chi \in S^1(\mathcal{H})$ we denote by $U_{\hat{\chi}}(\mathcal{H})$ the stabilizer subgroup of $\hat{\chi}$ w.r.t. the quotient action. $U_{\hat{\chi}}(\mathcal{H})$ is a closed subgroup of $U(\mathcal{H})$ and a Banach–Lie group with Lie algebra the subspace of anti-self-adjoint bounded operators commuting with the one dimensional projection operator P_χ on the ray generated by χ. In fact, $Lie(U_{\hat{\chi}}(\mathcal{H}))$ is a splitting subspace of $\mathfrak{u}(\mathcal{H})$, so that $U_{\hat{\chi}}(\mathcal{H})$ is a Lie subgroup of $U(\mathcal{H})$. If one changes the vector χ, one obtains a conjugate Lie group. Then by standard arguments $\mathbf{P}(\mathcal{H})$ is diffeomorphic to the orbit space $U(\mathcal{H})/U_{\hat{\chi}}(\mathcal{H})$ [17]. We also remark that the projection operator in $\mathfrak{u}(\mathcal{H})$ with range $Lie(U_{\hat{\chi}}(\mathcal{H}))$

$$A \mapsto P_\chi A P_\chi + (1 - P_\chi)A(1 - P_\chi)$$

is $Ad(U_{\hat{\chi}}(\mathcal{H}))$ invariant.

For $\chi \in S^1(\mathcal{H})$ one consider the symmetry S defined by $S = 1 - 2P_\chi$. Then one defines the involutive automorphism σ of $U(\mathcal{H})$ by the conjugation $A \mapsto SAS^{-1}$. The stability subgroup of σ is $U_{\hat{\chi}}(\mathcal{H})$. As a consequence, the quotient space $U(\mathcal{H})/U_{\hat{\chi}}(\mathcal{H})$ is a symmetric $U(\mathcal{H})$ space. One could identify $(\mathbf{P}(\mathcal{H}), U(\mathcal{H}), S)$ with $(U(\mathcal{H})/U_{\hat{\chi}}(\mathcal{H}), U(\mathcal{H}), S)$.

To the symmetric $U(\mathcal{H})$ space $(\mathbf{P}(\mathcal{H}), U(\mathcal{H}), S)$ it is associated the *symmetric Banach–Lie algebra* $(Lie(U(\mathcal{H})), Lie(U_{\hat{\chi}}(\mathcal{H}), \sigma))$, where

$$\mathfrak{g} \simeq Lie(U(\mathcal{H})) = \mathfrak{u}(\mathcal{H}), \qquad \mathfrak{h} \simeq Lie(U_{\hat{\chi}}(\mathcal{H})) = \mathfrak{u}(\mathcal{H}) \cap S',$$

i.e. the commutant of P_χ in $\mathfrak{u}(\mathcal{H})$.

Finally, $\mathfrak{m}$ is the anticommutant of P_χ in $\mathfrak{u}(\mathcal{H})$. We remark that σ is an involutive norm preserving Lie algebra automorphism.

We can define an $Ad(U_{\hat{\chi}}(\mathcal{H}))$ invariant scalar product in $\mathfrak{m}$ by

$$(A, B) := -\kappa \, Tr(AB).$$

The subspace $\mathfrak{m}$ is canonically identified with the tangent space at $\hat{\chi}$. Thus we get a Riemannian metric $\mathbf{g}$ on $\mathbf{P}(\mathcal{H})$.

The complex structure in $\chi^\perp$ induces an $Ad(U_{\hat{\chi}}(\mathcal{H}))$ invariant complex structure J on $\mathfrak{m}$. Thus a complex structure is induced on $\mathbf{P}(\mathcal{H})$. The proof that the metric $\mathbf{g}$ and the complex structure J correspond to the Fubini–Study metric and to the canonical complex structure of $\mathbf{P}(\mathcal{H})$ is a simple adaptation of the analogous finite dimensional statement [19]. $\square$

Corollary 1. *The projective space $\mathbf{P}(\mathcal{H})$ is connected and simply connected.*

Proof. This topological property is well known in the finite dimensional case [19]. In the infinite dimensional case it was proved by Kuiper [20] that the unitary group $U(\mathcal{H})$ is contractible. We have the exact sequence

$$\pi_1(U_{\hat{\chi}}(\mathcal{H})) \to \pi_1(U(\mathcal{H})) \to \pi_1(U(\mathcal{H})/U_{\hat{\chi}}(\mathcal{H})) \to \pi_0(U_{\hat{\chi}}(\mathcal{H}))$$
$$\to \pi_0(U(\mathcal{H})) \to (U(\mathcal{H})/U_{\hat{\chi}}(\mathcal{H})).$$

The exponential map $\mathfrak{u}(\mathcal{H}) \to U(\mathcal{H})$ is onto by a theorem of de la Harpe [15]. Therefore the isotropy subgroup $U_\chi(\mathcal{H})$ is connected. Thus by Proposition 1, we obtain that $\mathbf{P}(\mathcal{H})$ is connected and simply connected. $\square$

2.2. The Riemannian and Hermitian case

Let (M, G, s) be a symmetric homogeneous G space. We denote by H its stability subgroup. So we can identify M with the coset space G/H. We denote by o the equivalence class of e.

The *canonical connection* ∇ on M is defined by

$$\nabla_v(Y) := [X(v), Y]_x, \quad v \in T_x M$$

for all vector fields Y defined around x. This definition is consistent with the classical definition given by Kobayashi and Nomizu [19]. The canonical connection is invariant and complete.

We define, for $v \in \mathfrak{m}$ and $t \in \mathbf{R}$

$$c_v : t \mapsto = \exp(tv)o.$$

For every $v \in \mathfrak{m}$ the curve c_v is a geodesic starting from o of the canonical connection; conversely, every geodesic from o is of the form c_v for some $v \in \mathfrak{m}$.

Torsion and curvature of the canonical connection are discussed in [19]. Every symmetric homogeneous G space admits a unique torsion free G invariant affine connection, the *natural*

torsion free connection [19]. The natural torsion free connection has the same geodesics that the canonical connection.

Proposition 2. *If* (M, G, s) *is symmetric, there is a natural bijection between the set of all subspaces* $\mathfrak{m}'$ *of* $\mathfrak{m}$ *such that*

$$[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$$

and the set of all complete totally geodesic submanifolds M' *of* M (*through o*).

If (M, G, s) is symmetric Riemannian, then all symmetries are isometries and the canonical decomposition is orthogonal. Moreover, the canonical connection is the unique affine connection on M which is invariant w.r.t. all symmetries of M. We denote by T its torsion and by R its curvature.

Proposition 3. *The canonical connection satisfies*

1. $T = 0, \nabla R = 0$;

$$R(u, v)w = -[[u, v], w] \quad for \ u, v, w \in \mathfrak{m}.$$

2. *For every* $v \in \mathfrak{m}$ *the parallel transport along* $\pi(\exp(tv))$ *agrees with the differential of the transformation* $\exp(tv)$ *on M.*
3. *for every* $v \in \mathfrak{m}$

$$\pi(\exp(tv)) = \exp(tv)o$$

is a geodesic from o and conversely, every geodesic from o is of this form.
4. *Every G invariant tensor field on M is parallel.*

As a consequence, for every symmetric Riemannian G space $(M, G, s, \mathfrak{g})$ the canonical connection agrees with the natural torsion free connection. Moreover, the Riemannian metric on M induces the canonical connection.

A *Hermitian symmetric G space* is given by $(M, G, s, \mathfrak{g}, J)$, where $(M, G, s, \mathfrak{g})$ is a Riemannian symmetric G space and J is an almost complex structure on J, which is symmetry invariant. Then $\nabla J = 0$, J is integrable, so that (M, J) is a complex manifold.

Proposition 4. *Let* $(M, \mathfrak{g}, J)$ *with* J *almost complex and* $\mathfrak{g}$ *Hermitian metric. Then*:

1. *If* (M, G, s, J) *is a complex symmetric G space, then* J *is integrable and* $\mathfrak{g}$ *is Kähler.*
2. *If* $(\mathfrak{g}, J)$ *is a Kähler structure on a symmetric Riemann G space* $(M, G, s, \mathfrak{g})$, *then* $(M, G, s, \mathfrak{g}, J)$ *is Hermitian symmetric G space.*

We refer to $(M, \mathfrak{g}, J)$ as to the Kähler manifold underlying $(M, G, s, \mathfrak{g}, J)$.

Proposition 5. *Let* (M, G, s) *be a symmetric homogeneous G space, with isotropy group* H:

1. *If* m *admits some Ad(H) invariant complex structure I, then M admits an invariant complex structure such that the canonical connection is complex and M is complex affine symmetric.*
2. *If, moreover,* m *admits an Ad(H) invariant scalar product which is Hermitian w.r.t. I, then M admits an invariant Kähler metric and is Hermitian symmetric.*

We recall that a connected submanifold S of a Riemannian manifold M is *geodesic at* $m \in S$ if, for every $v \in T_m M$, the geodesic $c_{m,v}(t)$ determined by v lies in S for small values of the parameter t. If S is geodesic at every point of S, it is called a *totally geodesic submanifold* of M. By the above remarks we get that geodesics and hence totally geodesic submanifolds of a symmetric Riemannian space can be equivalently characterized as the geodesics and the totally geodesic submanifolds w.r.t. the canonical connection.

Closed complex totally geodesic submanifolds of a Hermitian symmetric G space corresponds to closed complex $Ad(H)$ invariant subspaces of m [19]. In the particular case of the projective space $\mathbf{P}(\mathcal{H})$, every closed complex subspace of m is $Ad(H)$ invariant. So we have the following proposition.

Proposition 6. *The projective space* $\mathbf{P}(\mathcal{H})$, *with the Fubini–Study metric, is the Kähler manifold underlying to the Hermitian symmetric space* $(\mathbf{P}(\mathcal{H}), U(\mathcal{H}), S, \mathbf{g}, J)$. *Closed complex totally geodesic submanifolds of* $\mathbf{P}(\mathcal{H})$ *at some point of* $\mathbf{P}(\mathcal{H})$ *correspond exactly to closed J invariant subspaces of* m.

We have shown that $\mathbf{P}(\mathcal{H})$ is a Hermitian symmetric G space. Hermitian symmetric G spaces M are naturally reductive and the canonical connection, the natural torsion free connection and the Riemannian connection agree [19]. Complete totally geodesic submanifolds through some point m in M correspond bijectively to $Ad(H)$ and J invariant closed subspaces of m [19]. In the particular case of $M = \mathbf{P}(\mathcal{H})$ one sees that $Ad(H)$ invariant subspaces of m are precisely the J invariant ones. Varying the point in $\mathbf{P}(\mathcal{H})$, we see that the family of all closed totally geodesic submanifolds of $\mathbf{P}(\mathcal{H})$ are identified with the family of all closed complex subspaces of $\mathcal{H}$. Of course, the manifold $\mathbf{P}(\mathcal{H})$ itself is considered as totally geodesic at any point. In this way we obtain a geometric interpretation of *Quantum Logic* $\mathcal{L}(\mathcal{H})$.

We stress that quantum logic has a relevant role in foundations of QM [22]. We can analogously prove that closed totally geodesic submanifolds of any Hermitian symmetric G space have the algebraic structure of a quantum logic. So it is very surprising and gratifying to see that this structure naturally appears in the general geometrical context of Hermitian symmetric G spaces.

3. Superposition principle and the geodesic structure of $\mathbf{P}(\mathcal{H})$

The projective space $\mathbf{P}(\mathcal{H})$ is metrically complete; the distance is

$$d(\hat{\varphi}, \hat{\psi}) = \sqrt{2\hbar} \arccos |(\varphi|\psi)|.$$

The diameter of $\mathbf{P}(\mathcal{H})$ is finite and equals $\sqrt{2\hbar}$. The *equator* of $\hat{\varphi}$ is the set of all $\hat{\psi}$ such that $d(\hat{\varphi}, \hat{\psi}) = \frac{1}{2} \operatorname{diam}(\mathbf{P}(\mathcal{H}))$. The *antipodal submanifold* $\hat{\varphi}^{\perp}$ of $\hat{\varphi}$ is the set of $\hat{\psi} \in \mathbf{P}(\mathcal{H})$ such that

$$d(\hat{\varphi}, \hat{\psi}) = \sqrt{2\hbar}.$$

We remark that the antipodal submanifold $\hat{\varphi}^{\perp}$ is the maximal closed complex totally geodesic submanifold of $\mathbf{P}(\mathcal{H})$ not containing $\hat{\varphi}$.

Antipodality has a very remarkable content: it translates into $\mathbf{P}(\mathcal{H})$ orthogonality:

1. $\hat{\varphi}, \hat{\psi} \in \mathbf{P}(\mathcal{H})$ antipodal if and only if $(\varphi|\psi) = 0$;
2. $\hat{\varphi}, \hat{\psi} \in \mathbf{P}(\mathcal{H})$ antipodal if and only if $d(\hat{\varphi}, \hat{\psi}) = \operatorname{diam}(\mathbf{P}(\mathcal{H}))$;
3. $\hat{\varphi}, \hat{\psi} \in \mathbf{P}(\mathcal{H})$ antipodal if and only if $\hat{\psi} \in C_{\hat{\varphi}}$,

with $C_{\hat{\varphi}}$ the *cut locus* of $\hat{\varphi}$, i.e. the complement of the greatest open neighborhood $U_{\hat{\varphi}}$ of $\hat{\varphi}$ such that any point of $U_{\hat{\varphi}}$ might be connected to $\hat{\varphi}$ by means of one and only one minimal geodesic.

For every $\hat{\varphi} \in \mathbf{P}(\mathcal{H})$, the exponential map

$$\operatorname{Exp}_{\hat{\varphi}} : T_{\hat{\varphi}}\mathbf{P}(\mathcal{H}) \to \mathbf{P}(\mathcal{H})$$

is defined on the whole $T_{\hat{\varphi}}\mathbf{P}(\mathcal{H})$ and the *injectivity radius*

$$R_{\hat{\varphi}}^{i} := \sup\{\rho > 0 | \operatorname{Exp}_{\hat{\varphi}} \lceil B(0_{\hat{\varphi}}; \rho) \text{ is injective}\}$$

($B(0_{\hat{\varphi}}; \rho)$ is the closed ball with radius ρ centered at $0_{\hat{\varphi}}$) is constant and equals $\hbar\pi$.

The mathematical formulation of Superposition Principle in the SQM is very well known (SP):

With the due care to normalization properties, superpositions mean $\mathbf{C}$-linear combinations.

Translation into $\mathbf{P}(\mathcal{H})$ is not particularly hard. *Statement* SP_1:

For any pair of distinct points $\hat{\varphi}, \hat{\psi}$ the set of all superpositions of these two states is $\mathbf{P}(\mathcal{H}_{\varphi,\psi})$, the projective of the two-dimensional subspace of $\mathcal{H}$ generated by any pair φ, ψ of representatives of $\hat{\varphi}, \hat{\psi}$.

For a sharp understanding of SP_1 one must supply a pointed geometrical characterization of $\mathbf{P}(\mathcal{H}_{\varphi,\psi})$ as a subset of $\mathbf{P}(\mathcal{H})$. This is done by looking at the geodesic structure of $\mathbf{P}(\mathcal{H})$. Let $v \in T_{\hat{\varphi}}\mathbf{P}(\mathcal{H})$, with normalized local representative $\xi \in \varphi^{\perp}$. The geodesic tangent in $\hat{\varphi}$ to v is

$$c_{\hat{\varphi},v}(t) = \mathbf{p}\left(\varphi \cos \frac{t}{\sqrt{2\hbar}} + \xi \sin \frac{t}{\sqrt{2\hbar}}\right),$$

where $\mathbf{p} : S^1(\mathcal{H}) \to \mathbf{P}(\mathcal{H})$ denotes the canonical surjection. In particular, $c_{\hat{\varphi},v}(\pi\sqrt{\hbar/2}) = \mathbf{p}(\xi)$, so that $\hat{\xi}$ is the (unique) antipodal point to $\hat{\varphi}$ lying on the geodesic $c_{\hat{\varphi},v}$.

More generally, if $\mathsf{v} = \rho\xi$ with $\|\mathsf{v}\| = \rho$ is a local representative of $v \in T_{\hat{\varphi}}\mathbf{P}(\mathcal{H})$, the geodesic $c_{\hat{\varphi},v}(t)$ is given by

$$c_{\hat{\varphi},v}(t) = \mathbf{p}\left(\varphi \cos \frac{\rho t}{\sqrt{2\hbar}} + \xi \sin \frac{\rho t}{\sqrt{2\hbar}}\right).$$

Now, using the geodesic structure of $\mathbf{P}(\mathcal{H})$, one easily sees that statement SP_1 is equivalent to the following *statement* SP_2:

For any pair $\hat{\varphi}$, $\hat{\psi}$ $(\hat{\varphi} \neq \hat{\psi})$ *the set of all superpositions of* $\hat{\varphi}$ *and* $\hat{\psi}$ *is the smallest totally geodesic submanifold of* $\mathbf{P}(\mathcal{H})$ *containing* $\hat{\varphi}$, $\hat{\psi}$.

For a complete geometric description of the physical content of superposition principle, we must be able to characterize *single* superpositions of states. We remark that ξ is a representative vector for the unique antipodal point to $\hat{\varphi}$ lying on the geodesic $c_{\hat{\varphi},v}$. A point $\hat{\chi}$ of $\mathbf{P}(\mathcal{H})$ belongs to $c_{\hat{\varphi},v}$ if and only if $\hat{\chi} \in \mathbf{P}(\mathcal{H}_{\varphi,\xi})$ and for some normalized vector $\varphi \in \hat{\varphi}$, one has

$$\frac{(\xi|\chi)}{(\varphi|\chi)} \in \mathbf{R}.$$

In particular, if ξ is a normalized representative for some antipodal point to $\hat{\varphi}$, then $\hat{\chi} = \widehat{\varphi + \xi}$ lies on the geodesic $c_{\hat{\varphi},v}$. Thus $\hat{\chi}$ lies on the intersection of the equator of $\hat{\varphi}$ with the geodesic $c_{\hat{\varphi},v}$. Conversely, this intersection point determines the geodesic $c_{\hat{\varphi},v}$. Any other point of the geodesic is the ray corresponding to some linear combination $\alpha\varphi + \beta\psi$ with real α and β. This intersection point is conveniently characterized as $b_\varphi(\psi)$ (or as $b_\psi(\varphi)$, since these rays are equal).

Geodesics connecting two antipodal points $\hat{\varphi}$ and $\hat{\psi}$ describe linear combinations $\alpha\varphi+\beta\psi$ with complex quotient α/β and are obtained altering the representative normalized vector for $\hat{\psi}$.

So we arrive to *statement* SP_3:

If φ, ψ *are orthogonal versors and* $\alpha, \beta \in \mathbf{C} - \{0\}$, *and*

$$\chi = \alpha\varphi + \beta\psi,$$

then

$$\hat{\chi} = c_{\hat{\varphi},v}\left(\arctan\left(\sqrt{2\hbar}\left|\frac{\beta}{\alpha}\right|\right)\right),$$

where the tangent vector v *corresponds in the chart* b_φ *to* $e^{i\theta}\psi$, *with* θ *denoting the relative phase* $\arg(\beta/\alpha)$ *of* α *and* β.

Therefore the full geometric formulation of the quantum superposition principle in $\mathbf{P}(\mathcal{H})$ *is given by* $SP_2 + SP_3$.

By the above discussion we see the physical relevance of the geodesic structure of the manifold of states. In general, we could conclude that for a purely "kinematical" formulation of the QSP we need, as a space of states, a manifold M equipped with a "*convenient*" geodesical structure. By the above discussion we conclude that such a convenient geodesical structure is provided by the structure of Hermitian symmetric G space. Of course, in this more general context, superpositions of two states are represented by the closed totally geodesic submanifold they generate.

Thinking of the well-known paper of Wick et al. [24] we can define a *superselection sector* of a Hermitian symmetric space M:

A superselection sector of M is a closed complex submanifold N of M such that for any pair $x \in N$, $y \in M - N$ *there is no geodesic connecting x with y.*

Thus superselection sectors are just connected components of M. Hence the projective space $\mathbf{P}(\mathcal{H})$ does not admit any not trivial superselection sector. We can, however, introduce superselection sectors on projective quantum mechanics by means of disjoint union of projective spaces.

4. Quantum superposition principle and observables

Let us remember, first of all, that *observables* of projective quantum mechanics are *real mean value maps* of bounded self-adjoint operators on $\mathcal{H}$, i.e. smooth maps $f : \mathbf{P}(\mathcal{H}) \to \mathbf{R}$ of the type $f(\hat{\varphi}) = \langle A \rangle_{\hat{\varphi}}$, where for $A \in B_{sa}(\mathcal{H})$ and $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$, we define $\langle A \rangle_{\hat{\varphi}} := (A\varphi | \varphi)$.

The map $\langle A \rangle$ is Hamiltonian, with Hamiltonian vector field $v_{\langle A \rangle}$ defined by

$$d_{\hat{\varphi}} \langle A \rangle (\xi) = \omega_{\hat{\varphi}}(v_{\langle A \rangle}(\hat{\varphi}), \xi) \quad \text{for } \xi \in \varphi^{\perp}.$$

A *Killing vector field* on a Riemannian manifold (M, g) is a complete vector field ξ whose flow preserves the Riemannian structure (i.e. $L_{\xi} \mathsf{g} = 0$). The following theorem was proved in [7].

Theorem 1. *A vector field ξ on $\mathbf{P}(\mathcal{H})$ is Killing if and only if there is a self-adjoint operator $A \in L(H)$ such that $\xi = v_{\langle A \rangle}$.*

A Hamiltonian function f on a Kähler manifold is said to be a *K function* if its Hamiltonian vector field v_f is Killing.

A smooth map $f : \mathbf{P}(\mathcal{H}) \to \mathbf{R}$ is *geolinear* if

$$f(c_{\hat{\varphi},v}(t)) = f(\hat{\varphi}) + \left(\sin \frac{t}{\sqrt{2\hbar}} \cos \frac{t}{\sqrt{2\hbar}} \right) d_{\hat{\varphi}} f(v) + \sin^2 \frac{t}{\sqrt{2\hbar}} Hess_{\hat{\varphi}} f(v, v)$$

($\hat{\varphi} \in \mathbf{P}(\mathcal{H})$, $v \in T_{\hat{\varphi}} \mathbf{P}(\mathcal{H})$ is a versor and $t \in \mathbf{R}$), where $c_{\hat{\varphi},v}$ is the geodesic through $\hat{\varphi}$ along v. We have proved in [7] the following theorem.

Theorem 2. *A map $f : \mathbf{P}(\mathcal{H}) \to \mathbf{R}$ is geolinear if and only if there is a self-adjoint operator $A \in L(\mathcal{H})$ such that $f = \langle A \rangle$.*

Thus the K functions on $\mathbf{P}(\mathcal{H})$ are precisely the geolinear maps and can be character-ized as functions preserving (in this particular sense) the superpositions and correspond to expectation value functions of bounded self-adjoint operators on $\mathcal{H}$.

We see from Theorems 1 and 2 that in projective quantum mechanics there is a strict link, as expected, between observables and the dynamical vector fields; but the really remarkable feature is the characterization of observables (as *geolinear maps*) and of dynamical vector fields (as *Killing vector fields*) with no more reference to mean value maps.

It could be of interest also to consider some Hamiltonian dynamics which not necessarily respect the Riemannian structure. So also non-linear observables and non-linear dynamic

evolutions could be suitably introduced. We will discuss this important point in a forth-coming paper [9]. Here we simply want to anticipate, as an example, some of such *flexible observables* on $\mathbf{P}(\mathcal{H})$.

We consider as example of a flexible observable the function

$$F := \langle A \rangle \langle B \rangle, \quad A, B \in B_{sa}(\mathcal{H}).$$

Since

$$(d_{\hat{\varphi}} \langle F \rangle)(w) = \langle B \rangle_{\hat{\varphi}} \omega(v_A(\hat{\varphi}), w) + \langle A \rangle_{\hat{\varphi}} \omega(v_B(\hat{\varphi}), w),$$

we have

$$v_{\langle F \rangle} = \langle A \rangle v_B + \langle B \rangle v_A.$$

We stress that by the above discussion we can conclude that *in principle it is possible to maintain the QSP in a non-linear QM provided the following conditions are respected*:

1. *the space of pure states is a symmetric Hermitian manifold (M, G, s, J, g);*
2. *the superpositions of $x, y \in M$, $(x \neq y)$, are the points of the smallest closed J invariant totally geodesic submanifold containing x and y;*
3. *the observables are those maps $f : M \to \mathbf{R}$ that preserve superpositions (the K functions);*
4. *the dynamical evolution is given by a vector field that preserves the Riemannian structure, i.e. by a Killing vector field on (M, g).*
   We could add:
   4.1. *the flexible dynamical evolution is given by Hamiltonian vector fields associated to some selected family of flexible observables.*

So we can introduce non-linear dynamics.

5. Uncertainty principle and Hermitian structure

In SQM for each observable $A \in B_{sa}(\mathcal{H})$ the *dispersion* in the "state" φ is introduced

$$\Delta_\varphi A := \|A\varphi - (\varphi|A\varphi)\varphi)\|, \quad \varphi \in S^1(\mathcal{H}),$$

and the Heisenberg uncertainty principle (HUP) is stated as follows.

Proposition 7. *For every $A, B \in B_{sa}(\mathcal{H})$ and every $\varphi \in S^1(\mathcal{H})$ the Heisenberg inequality holds:*

$$\Delta_\varphi A \cdot \Delta_\varphi B \geq \tfrac{1}{2} |(\varphi|[A, B]\varphi)|.$$

Therefore, in the natural Poisson structure [11–13] one has

$$\{\langle A \rangle, \langle B \rangle\} = \left\langle \frac{-\mathrm{i}}{\hbar}[A, B] \right\rangle,$$

$$\Delta_\varphi A = \sqrt{\tfrac{1}{2}\hbar} \|v_{\langle A\rangle}(\hat\varphi)\|_{\mathbf{g}},$$

and the Heisenberg inequality can be written as

$$|\{\langle A\rangle, \langle B\rangle\}(\hat\varphi)| \le \|v_{\langle A\rangle}(\hat\varphi)\|_{\mathbf{g}} \|v_{\langle B\rangle}(\hat\varphi)\|_{\mathbf{g}},$$

that is

$$|\omega_{\hat\varphi}(v_{\langle A\rangle}(\hat\varphi), v_{\langle B\rangle}(\hat\varphi))| \le \|v_{\langle A\rangle}(\hat\varphi)\|_{\mathbf{g}} \|v_{\langle B\rangle}(\hat\varphi)\|_{\mathbf{g}}.$$

Heisenberg inequality is nothing more than the uniform continuity of the symplectic form (or of the Poisson product) with respect to the topology induced on the tangent space by the Riemannian structure. The above argument also works for any pair of smooth functions, so we are lead outside of the realm of ordinary QM.

Rebus sic stantibus, we can say that the HUP can be formulated in a general setting which does not depend on the linearity properties of the setting.

Let M be a manifold endowed with a symplectic structure ω and a metric structure $\mathbf{g}$. We say that the HUP holds in $(M, \omega, \mathbf{g})$ if the symplectic form is uniformly continuous with respect to the topology of the tangent space induced by the metric, i.e. if

$$\exists a \in \mathbf{R}_+, \quad \text{such that } \forall x \in M, \qquad |\{f, h\}(x)| \le a\|v_f(x)\|_{\mathbf{g}} \|v_h(x)\|_{\mathbf{g}} \quad \text{(HUP)}$$

for any pair of Hamiltonian functions f, h (with Hamiltonian vector field v_f and v_h, respectively). Indeed, if HUP holds and we define

$$\Delta_x f := \sqrt{\tfrac{1}{2}r} \|v_f(x)\|_{\mathbf{g}},$$

where

$$r := \min\{a \in \mathbf{R}_+ \text{ such that HUP holds}\},$$

then

$$\Delta_x f \Delta_x h \ge \tfrac{1}{2}|\{f, h\}(x)|.$$

One could also introduce the dispersion function for a field X by

$$\Delta_x X := \sqrt{\tfrac{1}{2}r}(\mathbf{g}_x(X_x, X_x))^{1/2}$$

getting

$$\Delta_x X \Delta_x Y \ge \tfrac{1}{2}r|\omega_x(X_x, Y_x)|.$$

Therefore, adding a fifth requirement to the requirements (1)–(4) above, we can draw the conclusion.

In principle it is possible to maintain the QSP and the HUP in a non-linear quantum mechanics assuming (1)–(4) as above and

5. *the Riemannian manifold $(M, \mathbf{g})$ of pure states is endowed with a symplectic form ω which is uniformly continuous with respect to $\mathbf{g}$.*

In particular, this holds in any Hermitian symmetric space M, but we can say more. Actually, HUP holds with $r = 1$ since the complex structure operator is unitary. Moreover, the set of Killing vector fields of M is *full*, i.e. every tangent vector v at x belongs to some Killing vector field, for every $x \in M$. To see this, given $y \in M$, choose some $g \in G$ such that $y = gx$. Then define $\xi_y := g_*(v)$. One easily verifies that ξ is a (well defined) Killing vector field.

In any Hermitian symmetric space M, HUP can be stated in a strong form.

Proposition 8. *If M is a Hermitian symmetric space, then for every vector field X and every $x \in M$ there exists a Killing vector field K such that*

$$\Delta_x X \Delta_x K = |\omega_x(X_x, K_x)|.$$

Proof. We can assume $\Delta_x X \neq 0$. We know by HUP that for every vector field Y there exists $\lambda \in \mathbf{R}_+$ such that

$$\lambda^2 \mathsf{g}_x(Y_x, Y_x) \geq (\omega_x(X_x, Y_x))^2.$$

In particular, this holds for any Killing vector field K such that

$$K_x = Y_x = J_x X_x.$$

We obtain

$$\lambda^2 \mathsf{g}_x(K_x, K_x) = \lambda^2 \mathsf{g}_x(X_x, X_x) \geq (\omega_x(X_x, J_x X_x))^2 = (\mathsf{g}_x(X_x, X_x))^2,$$

so that

$$\lambda^2 \geq \mathsf{g}_x(X_x, X_x) = 2\Delta_x^2 X$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box$

A natural physical requirement is that all Killing vector fields are Hamiltonian. This is true if M is simply connected. In every simply connected Hermitian symmetric space M, the set of K functions is *full*, i.e. the differentials of K functions span the whole cotangent space T_x^*M, for every $x \in M$. This implies that for every K function f and $x \in M$ there exists a K function h such that

$$|\{f, h\}(x)| = \|v_f(x)\|_{\mathsf{g}} \|v_h(x)\|_{\mathsf{g}}$$

(see Proposition 4.5 in [11]).

The Cartan Ambrose Hicks theorem allows one to characterize simply connected symmetric (complex) manifolds as (complex) Banach manifolds admitting a geodesically complete torsion free affine connection whose curvature tensor is parallel [25].

The dispersion function is well defined also for non-geolinear functions. In particular, for a flexible observable we have

$$\Delta_{\hat{\varphi}}^2(\langle F \rangle) = \langle B \rangle_{\hat{\varphi}}^2 \Delta_{\hat{\varphi}}^2(\langle A \rangle) + \langle A \rangle_{\hat{\varphi}}^2 \Delta_{\hat{\varphi}}^2(B) + 2\langle A \rangle_{\hat{\varphi}} \langle B \rangle_{\hat{\varphi}} \langle A \circ B \rangle_{\hat{\varphi}},$$

where $A \circ B$ denotes the Jordan product. In particular, if A and B commute

$$\Delta_{\hat{\varphi}}^2(\langle F \rangle) = \langle B \rangle_{\hat{\varphi}}^2 \Delta_{\hat{\varphi}}^2(\langle A \rangle) + \langle A \rangle_{\hat{\varphi}}^2 \Delta_{\hat{\varphi}}^2(B) + 2\langle A \rangle_{\hat{\varphi}} \langle B \rangle_{\hat{\varphi}} \langle AB \rangle_{\hat{\varphi}}.$$

6. Spectral theory and quantum probability principle

In this section, we show that ordinary spectral theory for self-adjoint operators can be easy recovered by the corresponding Killing vector fields and the dispersion function. But it has to be stressed that this formulation works very well also for *non-Killing* vector fields. This opens the possibility to found a non-linear spectral theory. For previous attempts in this direction, see [3].

We can define the spectrum of $\langle A \rangle$ for $A \in B_{sa}(\mathcal{H})$. We say that:

1. $\lambda \in \mathbf{R}$ is a *regular value* if

$$\exists \epsilon > 0 \quad \text{such that} \quad \langle (A - \lambda)^2 \rangle_{\hat{\varphi}} > \epsilon \quad \forall \hat{\varphi} \in \mathbf{P}(\mathcal{H});$$

2. λ is a *eigenvalue* if

$$\langle (A - \lambda)^2 \rangle_{\hat{\varphi}} = 0 \quad \text{for some} \ \ \hat{\varphi} \in \mathbf{P}(\mathcal{H});$$

3. λ *belongs to continuous spectrum* if

$$\langle (A - \lambda)^2 \rangle_{\hat{\varphi}} > 0 \quad \forall \hat{\varphi} \in \mathbf{P}(\mathcal{H}), \quad \text{and} \quad \exists \{\hat{\varphi}_n\} \quad \text{such that} \ \ \langle (A - \lambda)^2 \rangle_{\hat{\varphi}_n} \to 0.$$

This definition of spectrum agrees with the standard one for $A \in B_{sa}(\mathcal{H})$. However, this definition immediately extends to *every Hamiltonian function* on $\mathbf{P}(\mathcal{H})$.

Now we discuss spectral aspects in terms of the Hamiltonian vector field $v_{\langle A \rangle}$.

First, we remark that

$$v_{\langle (A-\lambda) \rangle} = v_{\langle A \rangle},$$

so that the Hamiltonian vector fields, alone, do not allows to characterize the spectral points. However, since

$$\hbar \| \mathsf{v}_{\langle A \rangle}(0) \| = \| A\varphi - (A\varphi|\varphi)\varphi \|,$$

we get that $v_{\langle A \rangle}(\hat{\varphi}) = 0$ if and only if

$$A\varphi = \lambda\varphi \quad \text{with} \ \ \lambda = (A\varphi|\varphi).$$

To get eigenvectors of A consider those versors φ such that $v_{\langle A \rangle}(\hat{\varphi}) = 0$; the corresponding eigenvalue is given by $\langle A \rangle_{\hat{\varphi}}$.

We can also characterize the points of continuous spectrum. A $\lambda \in \mathbf{R}$ belongs to the spectrum of A if and only if for every $\epsilon > 0$ there exists a versor φ such that

$$\| (A - \lambda)\varphi \| < \epsilon.$$

This implies that there exists a sequence $\{\varphi_n\}$ of versors such that

$$\lim_n (A\varphi_n|\varphi_n) = \lambda,$$

so that

$$\|(A - (A\varphi_n|\varphi_n))\varphi_n\| \to 0.$$

This means that the sequence $\{(A\varphi_n|\varphi_n)\}$ is Cauchy and that $\Delta_{\varphi_n} A \to 0$. The last condition amounts to the request that the local expression of the field $v_{\langle A \rangle}(\hat\varphi_n)$ in the chart b_{φ_n} goes to 0 for $n \to \infty$.

Conversely, let $\lambda \in \mathbf{R}$ such that for some sequence $\{\hat\varphi_n\}$

$$\langle A \rangle_{\hat\varphi_n} \to \lambda, \qquad \Delta_{\varphi_n} A \to 0$$

then λ belongs to the spectrum of A; if, moreover, it does not exists any $\hat\varphi$ such that

$$\langle A \rangle_{\hat\varphi} = \lambda, \qquad v_{\langle A \rangle}(\hat\varphi) = 0$$

then λ belongs to the continuous spectrum of A.

We also remark that

$$\hbar \|\mathbf{v}_{\langle A \rangle}(0)\|^2 = \|A\varphi - (\varphi|A\varphi)\varphi\|^2 = \langle A^2 \rangle_{\hat\varphi} - \langle A \rangle_{\hat\varphi}^2.$$

We conclude that λ belongs to the spectrum of A if and only if there exists a sequence $\{\hat\varphi_n\}$ such that

$$\langle A \rangle_{\hat\varphi_n} \to \lambda, \qquad \langle A^2 \rangle_{\hat\varphi_n} \to \lambda^2.$$

For a given linear operator A defined on $\mathcal{H}$, $A \neq \mathbf{0}$, we define the *regularity domain* of A as the open set

$$\mathbf{P}(\mathcal{H}) - \mathbf{P}(Ker\, A) = \{\hat\varphi \in \mathbf{P}(\mathcal{H})|A\varphi \neq 0\}.$$

We observe that A quotients to a transformation

$$\hat{A} : \mathcal{D}_{\hat{A}} \to \mathbf{P}(\mathcal{H}), \qquad \hat{A}\hat\varphi := \widehat{A\varphi},$$

where $\mathcal{D}_{\hat{A}} := \mathbf{P}(\mathcal{H}) - \mathbf{P}(Ker\, A)$. The transformation $\hat{A}$ is smooth if and only if $A \in B(\mathcal{H})$.

We can use regularity domains to characterize the spectra of bounded self-adjoint operators. Let $A \in B_{sa}(\mathcal{H})$. Then $\lambda \in \mathbf{R}$ is said to be *regular value for A* if:

1. $\mathcal{D}_{\widehat{(A-\lambda)}} = \mathbf{P}(\mathcal{H})$;
2. $\widehat{(A - \lambda)}$ is a diffeomorphism.

We say that λ is a *spectral value for A* if it is not a regular value.
This means that either

1. $\mathcal{D}_{\widehat{(A-\lambda)}} \neq \mathbf{P}(\mathcal{H})$;
    or
2. $\mathcal{D}_{\widehat{(A-\lambda)}} = \mathbf{P}(\mathcal{H})$ but $\widehat{(A - \lambda)}$ is not a diffeomorphism.

We remark that $\widehat{(A - \lambda)}$ is a smooth bijection, but its inverse can fail to be smooth. Spectral values λ of type (1) are said to be *eigenvalues of $\hat{A}$*. Spectral values of type (2) are

said to belong to the *continuous spectrum of* $\hat{A}$. This definition of spectrum agrees with the precedent one.

Remark. We know that $(A - \lambda)^{-1}$, if it exists, is a linear operator defined on all the Hilbert space $\mathcal{H}$, therefore the quotient map $\widehat{(A - \lambda)^{-1}}$ exists and agrees with $\widehat{(A - \lambda)}^{-1}$; hence $\widehat{(A - \lambda)}^{-1}$ is smooth if and only if $(A - \lambda)^{-1} \in B(\mathcal{H})$.

Coming to the probabilistic interpretation, we remember that in SQM the following rule is posited:

The probability that a measurement of the observable A (s.a. operator of $\mathcal{H}$) in the state W (von Neumann density operator on $\mathcal{H}$) gives an outcome in a Borel set X on **R** *is given by*

$$P(A, W, X) = Tr(W Q^A(X)),$$

where Q^A is the spectral measure of A.

A *quantum probability measure* for $\mathcal{H}$ is a map $\mu : \mathcal{L}(\mathcal{H}) \to \mathbf{R}_+$ such that $\mu(\mathbf{1}) = 1$ and

$$\mu(P + Q) = \mu(P) + \mu(Q),$$

whenever P and Q are orthogonal. Remember that $\mathcal{L}(\mathcal{H})$ denotes the quantum logic of $\mathcal{H}$.

If, whenever $\{P_i\}_{i \in I}$ is a family of mutually orthogonal projections, $\sum_i \mu(P_i)$ is convergent and

$$\mu\left(\sum_i P_i\right) = \sum_i \mu(P_i),$$

then μ is said to be *completely additive*. Completely additive quantum probability measures form a convex set, *the set of states of $\mathcal{H}$*.

To every ray $\hat{\varphi}$ we can associate the projection operator on the ray $P_{\hat{\varphi}}$. Then the map

$$\mu_{\hat{\varphi}} : \mathcal{L}(\mathcal{H}) \to \mathbf{R}, \qquad Q \mapsto \mu_{\hat{\varphi}}(Q) := Tr(P_{\hat{\varphi}} Q)$$

is a completely additive quantum probability measure. Moreover, $\mu_{\hat{\varphi}}$ is pure, i.e. cannot be not trivially expressed as convex combination of quantum probability measures.

The essential content of Gleason Theorem is that every completely additive quantum probability measure μ on $\mathcal{L}(\mathcal{H})$ has a unique extension to a positive normal functional Φ_μ on $B(\mathcal{H})$, whenever $dim(\mathcal{H}) > 2$. This implies that there exists a unique positive, self-adjoint trace class operator (*density operator*) W such that $Tr\, W = 1$ and

$$\Phi_\mu(A) = Tr(WA) \quad \forall A \in B(\mathcal{H}).$$

Equivalently,

$$\mu(Q) = Tr(WQ)$$

for every projection operator Q. In particular, Φ_μ is pure if and only if W is a one-dimensional projection operator, i.e. if $\mu = \mu_{\hat{\varphi}}$ for some $\hat{\varphi} \in \mathbf{P}(\mathcal{H})$.

Therefore, mixed states can be interpreted as probability measures on the family of closed totally geodesic submanifolds of $\mathbf{P}(\mathcal{H})$. Elements of $\mathbf{P}(\mathcal{H})$ (pure states) corresponds precisely to pure probability measures.

It is well known by spectral theory of density operators that functionals Φ_μ can be uniquely expressed as probability measures on $\mathbf{P}(\mathcal{H})$, or else as σ convex combinations of pure states.

We are able to characterize the trace functional and probability transition map in terms of the metric structure of $\mathbf{P}(\mathcal{H})$. In the projective space totally geodesic submanifolds M correspond to projection operators Q_M on the Hilbert space $\mathcal{H}$, with the property that M is canonically identified with the projective space of the range of Q_M. Then $\langle Q_M \rangle$ is the unique geolinear map such that

$$\langle Q_M \rangle_{\hat{\varphi}} = 1 \quad \forall \hat{\varphi} \in M, \qquad \langle Q_M \rangle_{\hat{\varphi}} = 0 \quad \forall \hat{\varphi} \in M^\perp = \mathbf{P}(Ker\, Q_M).$$

In particular, there is a unique geolinear map $\langle Q_{\hat{\varphi}} \rangle$ such that

$$\langle Q_{\hat{\varphi}} \rangle_{\hat{\varphi}} = 1, \qquad \langle Q_{\hat{\varphi}} \rangle_{\hat{\psi}} = 0 \quad \text{for } \hat{\psi} \in \hat{\varphi}^\perp.$$

So we obtain the probability transition map

$$\langle \hat{\varphi} | \hat{\psi} \rangle := \langle Q_{\hat{\varphi}} \rangle_{\hat{\psi}}.$$

So traces and probability transitions are obtained as the corresponding K functions: for $\hat{\varphi} \in \mathbf{P}(\mathcal{H})$

$$\langle Q_M \rangle_{\hat{\varphi}} = Tr(Q_M P_{\hat{\varphi}}) = \sqrt{2\hbar} \inf_{\hat{\psi} \in M} \arccos |(\varphi | \psi)| = d(\hat{\varphi}, M).$$

Therefore, for a pure state $\mu = \mu_{\hat{\varphi}}$, we have

$$P(A, P_{\hat{\varphi}}, X) = \mu_{\hat{\varphi}}(Q^A(X)) = d(\hat{\varphi}, M^A(X)),$$

where $M^A(X)$ is the totally geodesic submanifold of $\mathbf{P}(\mathcal{H})$ canonically associated to the projective space of the range of the projection operator $Q^A(X)$, for a given Borel set X.

The submanifold $M^A(X)$ can be characterized as the unique closed totally geodesic submanifold of $\mathbf{P}(\mathcal{H})$ such that

$$\langle A \rangle_{\hat{\varphi}} \in X \quad \text{for } \hat{\varphi} \in M^A(X), \qquad \langle A \rangle_{\hat{\varphi}} \in \mathbf{R} - X \quad \text{for } \hat{\varphi} \in (M^A(X))^\perp.$$

Every mixed state Φ_μ, associated to some density operator W is a σ convex combination of pure states $\Phi_\mu = \sum_i \alpha_i \Phi_{\mu_i}$ with $\mu_i = \mu_{\hat{\varphi}_i}$. Therefore, we obtain

$$P(A, W, X) = \sum_i \alpha_i d(\hat{\varphi}_i, M^A(X)).$$

This gives the geometric content of the probabilistic interpretation.

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References

[1] M.C. Abbati, R. Cirelli, A. Manià, The orbit space of the action of gauge transformation group on connections, J. Geom. Phys. 6 (1989) 537–557.

[2] I. Antoniou, E. Gunzig, P. Nardone, G.P. Pronko, A Hamiltonian formalism for classical systems interacting with quantum systems, Mod. Phys. Lett. A 14 (33) (1999) 2287–2302.

[3] A. Ashtekar, T.A. Shilling, Geometrical formulation of quantum mechanics, gr-qc/9706069, 1997.

[4] A.L. Besse, Einstein Manifolds, Erg. der Math., Vol. 3, Band 10, Springer, Berlin, 1987.

[5] P. Bóna, Extended quantum mechanics, Acta Phys. Slovaca 50 (1) (2000) 1–198.

[6] D.C. Brody, L.P. Hughston, Geometric quantum mechanics, quant-ph/9906086, 1999.

[7] R. Cirelli, M. Gatti, A. Manià, On the non-linear extension of quantum superposition and uncertainty principles, J. Geom. Phys. 29 (1999) 64–86.

[8] R. Cirelli, M. Gatti, A. Manià, Fundamental principles of quantum mechanics and non-linearity, in: H.D. Doebner, S.T. Ali, M. Keyl, R.F. Werner (Eds.), Trends in Quantum Mechanics, Proceedings of the International Symposium on Goslar, 1998, World Scientific, Singapore, 2000.

[9] R. Cirelli, M. Gatti, A. Manià, Flexible non-linear observables and dynamics in quantum mechanics, in preparation.

[10] R. Cirelli, P. Lanzavecchia, A. Manià, Normal pure states of the Von Neumann algebra of bounded operators as Kähler manifold, J. Phys. A 16 (1983) 3829.

[11] R. Cirelli, A. Manià, L. Pizzocchero, Quantum mechanics as an infinite-dimensional Hamiltonian system with uncertainty structure: Part I, J. Math. Phys. 31 (12) (1990) 2891–2897.

[12] R. Cirelli, A. Manià, L. Pizzocchero, Quantum mechanics as an infinite-dimensional Hamiltonian system with uncertainty structure: Part II, J. Math. Phys. 31 (12) (1990) 2898–2903.

[13] R. Cirelli, A. Manià, L. Pizzocchero, Quantum phase space formulation of Schrödinger mechanics, Int. J. Mod. Phys. A 6 (1991) 2133–2146.

[14] R. Cirelli, A. Manià, L. Pizzocchero, A functional representation for non-commutative C^* algebras, Rev. Math. Phys. 6 (5) (1994) 675–697.

[15] P. de la Harpe, Classical Banach–Lie Algebras and Banach–Lie Groups of Operators in Hilbert Space, Lecture Notes in Mathematics, Vol. 285, Springer, Berlin, 1972.

[16] M. Gatti, Sull'estensione non-lineare dei principi fondamentali della meccanica quantistica, Ph.D. Thesis, Università Statale di Milano, 1997.

[17] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

[18] T.W.B. Kibble, Relativistic models of nonlinear quantum mechanics, Commun. Math. Phys. 64 (1978) 73–82.

[19] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vols. I and II, Wiley, New York, 1969.

[20] N.H. Kuiper, The homotopy type of the unitary group of Hilbert space, Topology 3 (1965) 19–30.

[21] T.F. Jordan, Reconstructing a nonlinear dynamical framework for testing quantum mechanics, Ann. Phys. 225 (1993) 83–113.

[22] G. Mackey, The Mathematical Foundations of Quantum Mechanics, Benjamin, New York, 1963.

[23] S. Weinberg, Testing quantum mechanics, Ann. Phys. 194 (1989) 336.

[24] G.C. Wick, A.S. Wightman, E.P. Wigner, The intrinsic parity of elementary particles, Phys. Rev. 88 (1952) 101–105.

[25] D.R. Wilkins, Infinite dimensional homogeneous manifolds, Proc. R. Ir. Acad. A 94 (1) (1994) 105–118.